New Equations for Gravitation with the Riemann Tensor and 4-Index Energy-Momentum Tensors for Gravitation and Matter*

Zahid Zakir†
Centre for Theoretical Physics and Astrophysics,
P.O.Box 4412, Tashkent 700000 Uzbekistan

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Abstract

A generalized version of the Einstein equations in the 4-index form, containing the Riemann curvature tensor linearly, is derived. It is shown, that the gravitational energy-momentum density outside a source is represented across the Weyl tensor vanishing at the 2-index contraction. The 4-index energy-momentum density tensor for matter also is constructed.

1 Introduction

The definition of the energy-momentum for the gravitational field is a more complicated procedure then for matter [1]. Due to the vanishing of the Einstein tensor G_{ik} outside a source:

$$G_{ik} = \frac{1}{\kappa}(R_{ik} - \frac{1}{2}g_{ik}R) = 0, \tag{1}$$

one can not treat it directly as the energy-momentum density for gravitation.

The standard methods of asymptotically flat space-time become meaningful only at large distances from the source. Particularly, the introduction of pseudotensors leads to difficulties which have not been overcame. Even the such rigorous treatment of the gravitational energy as the Hamiltonian formulation leads to the conclusion about its non-localizability due to non-tensor character of their observables (see [2]).

At the same time, the curvature tensor R_{iklm} more than all other observables can be considered as a true characteristics of gravity. For this reason, the explanation of the statement about non-localizability of the gravitational energy by the principle of equivalence is

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†E-mail: zahid@in.edu.uz

incorrect because of the local and covariant character of the curvature tensor, appeared in the theory also due to that principle.

In the present paper a generalized version of the Einstein equations directly containing the Riemann tensor will be formulated. Really, the tensor G_{ik}, vanishing in empty space, in fact is the contraction of the nonvanishing 4-index tensor:

$$g^{il}G_{iklm} = G_{km} = 0. \tag{2}$$

In the paper we shall derive G_{iklm} from the Einstein-Hilbert action function as:

$$G_{iklm} = \frac{1}{\kappa}\left[R_{iklm} - \frac{1}{2(d-1)}(g_{il}g_{km} - g_{im}g_{kl})R\right], \tag{3}$$

where d is the spacetime dimensionality, and then obtain 4-index gravitational equations:

$$G_{iklm} = T_{iklm}. \tag{4}$$

Here the 4-index energy-momentum density tensor of the source T_{iklm} is defined as:

$$T_{iklm} = V_{iklm} + T^{(m)}_{iklm}, \tag{5}$$

where $T^{(m)}_{iklm}$ is appropriately symmetrized a combination of the standard energy-momentum density tensor of matter T_{ik} and its scalar T. Here V_{iklm} is the new truly 4-index energy-momentum density tensor with the property $g^{il}V_{iklm} = 0$, which does not vanish outside the source (in the vacuum) and, therefore, can be interpreted as *the energy-momentum density of the gravitational field.*

In the vacuum G_{iklm} contains only the Weyl tensor C_{iklm} and we have the equations for the gravitational field:

$$\frac{1}{\kappa}C_{iklm} = V_{iklm}. \tag{6}$$

2 The action function and field equations with the Riemann tensor

The gravitational equations we obtain from the Einstein-Hilbert action function:

$$S = \int d\Omega\sqrt{-g}\left(-\frac{1}{2\kappa}R + L\right), \tag{7}$$

where L is the matter Lagrangian. The variation of the geometric term can be represented as:

$$\delta S_g = -\frac{1}{4\kappa}\delta_g \int d\Omega\sqrt{-g}(g^{il}g^{km} - g^{im}g^{kl})R_{iklm}. \tag{8}$$

The relationships:

$$g^{il}\delta R_{il} = g^{il}\delta(g^{km}R_{iklm}) = g^{il}\delta g^{km}R_{iklm} + g^{il}g^{km}\delta R_{iklm}, \tag{9}$$

allows one rewrite the variations of δR_{iklm} in the form:

$$g^{il}g^{km}\delta R_{iklm} = -g^{il}\delta g^{km}R_{iklm} + g^{il}\delta R_{il}. \tag{10}$$

Then, by taking into account the formulae:

$$\delta(\sqrt{-g})R = -\frac{1}{2}\sqrt{-g}\delta g^{km}g_{km}R = \tag{11}$$

$$= \sqrt{-g}\delta g^{km}g^{il}\left[-\frac{1}{2(d-1)}(g_{km}g_{il} - g_{kl}g_{im})R\right], \tag{12}$$

we obtain:

$$\delta S_g = -\frac{1}{2\kappa}\int d\Omega\sqrt{-g}(G_{iklm}g^{il})\delta g^{km}, \tag{13}$$

where G_{iklm} is presented in Eq.(3).

The Riemann tensor can be represented as:

$$R_{iklm} = C_{iklm} + \frac{1}{(d-2)}(g_{km}R_{il} - g_{kl}R_{im} + g_{il}R_{km} - g_{im}R_{kl}) + \\ -\frac{1}{(d-1)(d-2)}(g_{il}g_{km} - g_{im}g_{kl})R, \tag{14}$$

where d is the spacetime dimensionality, and C_{iklm} has the property $g^{il}C_{iklm} = 0$. If $C_{iklm} = 0$, the such manifold is conformally flat.

We introduce a corresponding 4-index energy-momentum density tensor for the source with the symmetry properties the same as for R_{iklm} as:

$$T_{iklm} = V_{iklm} + T^{(m)}_{iklm}. \tag{15}$$

Here V_{iklm} is the truly 4-index part of the energy-momentum density tensor of the source with the vanishing contraction, and $T^{(m)}_{iklm}$ is constructed from the 2-index energy-momentum density tensor of matter T_{km} as:

$$T^{(m)}_{iklm} = \frac{1}{(d-2)}(g_{km}T_{il} - g_{kl}T_{im} + g_{il}T_{km} - g_{im}T_{kl}) - \\ -\frac{T}{(d-1)(d-2)}(g_{il}g_{km} - g_{im}g_{kl}), \tag{16}$$

$$T = g^{km}T_{km} = \frac{1}{2}(g^{il}g^{km} - g^{im}g^{kl})T_{iklm}. \tag{17}$$

Then, for the variation of source's action function we have:

$$\delta_g S_m = \frac{1}{2}\int d\Omega\sqrt{-g}\delta g^{km}g^{il}T_{iklm}. \tag{18}$$

The result of the variational procedure, therefore, is:

$$\delta S = -\frac{1}{2}\int d\Omega\sqrt{-g}\delta g^{km}g^{il}\left(G_{iklm} - T_{iklm}\right). \tag{19}$$

which gives the field equations:

$$g^{il}(G_{iklm} - T_{iklm}) = 0. \tag{20}$$

Therefore, we obtain 4-index equations for the gravitational field in the form:

$$G_{iklm} = T_{iklm}, \tag{21}$$

or:

$$\frac{1}{\kappa}\left[R_{iklm} - \frac{1}{2(d-1)}(g_{il}g_{km} - g_{im}g_{kl})R\right] = T_{iklm}. \tag{22}$$

The covariant derivatives of these 4-index tensors in the case $d = 4$ are:

$$\begin{aligned} G^{i}_{.klm;i} &= \frac{1}{\kappa}\left[R^{i}_{.klm;i} - \frac{1}{6}(g_{km}R_{,l} - g_{kl}R_{,m})\right] = \\ &= T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{,l} - g_{kl}T_{,m}), \quad (23) \\ T^{j(m)}_{klm;j} &= \frac{1}{2}\left[T_{km;l} - T_{kl;m} - \frac{1}{3}(g_{km}T_{;l} - g_{kl}T_{;m})\right] = \frac{1}{2}G^{i}_{.klm;i}. \quad (24) \end{aligned}$$

Then we obtain the relationship:

$$V^{j}_{klm;j} = G^{j}_{klm;j} - T^{j(m)}_{klm;j} = \frac{1}{2}G^{i}_{.klm;i}. \tag{25}$$

and, therefore,

$$V^{j}_{klm;j} = T^{j(m)}_{klm;j} \tag{26}$$

In the vacuum, therefore, there are local conservation laws:

$$G^{j}_{.klm;j} = V^{j}_{klm;j} = 0. \tag{27}$$

3 Conclusions

The very important part of the Riemann tensor is the Weyl tensor C_{iklm} which, in fact, defines the gravitational field outside the source. This part of the Riemann tensor ordinarily has been lost at the 2-index contraction and this was a reason for the difficulties with the definition of the gravitational energy.

It is shown that the 4-index energy-momentum tensor for matter must contain the additional pure 4-index term V_{iklm} which has all required properties of the energy-momentum tensor for the gravitational field. The discussion of some applications of this new treatment of the gravitational energy and its connections with other definitions will be presented in [3].

References

[1] Einstein A. (1918) Sitz.preuss.Akad.Wiss. v.1, p.448.

[2] Faddeev L.D. (1982) Uspekhi, **136**, 435.

[3] Zakir Z. (1999) *Four-Index Equations for Gravitation and the Gravitational Energy-Momentum Tensor.* (gr-qc/9906039), 10 p., in "Z.Zakir (2003) *Structure of Space-Time and Matter*. CTPA, Tashkent".